# On The Dimensionality of Bounds Generated by the Shapley-Folkman Theorem

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**Abstract**

The Shapley-Folkman theorem places a scalar upper bound on the distance between a sum of non-convex sets and its convex hull. We observe that some information is lost when a vector is converted to a scalar to generate this bound and propose a simple normalization of the underlying space which mitigates this loss of information. As an example, we apply this result to the Anderson (1978) core convergence theorem, and demonstrate how our normalization leads to an intuitive, unitless upper bound on the discrepancy between an arbitrary core allocation and the corresponding competitive equilibrium allocation.

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*Keywords:* Shapley-Folkman Theorem, General Equilibrium Theory

## 1. Introduction

The Shapley-Folkman theorem places an upper bound on the distance between a sum of non-convex sets in $\mathbb{R}^N$ and its convex hull. Roughly speaking, the theorem states that the distance between a sum of sets in $\mathbb{R}^N$ and its convex hull can be no larger than the N largest summands. Thus, as the number of sets becomes arbitrarily large, the distance between the convex hull of the sum and the sum itself becomes proportionately negligible. One notable application of this theorem is Anderson's (1978) core convergence result, which states that under very general conditions, a measure of the discrepancy between a core allocation and the corresponding competitive equilibrium price vector in a pure exchange economy becomes arbitrarily small as the number of agents gets large.

Our key observation is that applications of the Shapley-Folkman theorem can be sensitive to the choice of units in which the underlying commodities are measured, and, as such, there may be a significant loss of information when an N-dimensional vector is converted to a scalar as required by the theorem. We propose a simple normalization which mitigates this loss of information, and we restate the Shapley-Folkman and Anderson theorems after the normalization has been performed. In addition, we suggest an alternative, unitless measure of convergence which immediately follows from the Anderson result and prove that our normalization generates a least upper bound with respect to this measure.

The structure of the paper is organized as follows: Section 2 states the Shapley-Folkman theorem, along with one of our key results, a corollary which incorporates our proposed normalization; Section 3 states the Anderson (1978) theorem and presents some results applying our normalization in that context; and Section 4 concludes.

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## 2. The Shapley-Folkman Theorem

Our statement of the Shapley-Folkman theorem is taken directly from Starr (2008) and reproduced here for the reader's convenience. Before stating the theorem, it is necessary to introduce a bit of notation.[1] Let $S$ be a compact subset of $\mathbb{R}^N$, and let con$(S)$ be defined as the closed convex hull of $S$ (the smallest closed convex set containing $S$ as a subset). A simple example of a non-convex set in $\mathbb{R}^2$ would be a disk with an indentation or hole. The convex hull of this set would "fill in" the hole. Then define

$$\text{rad}(S) \equiv \inf_{x \in \mathbb{R}^N} \sup_{y \in S} |x - y|.$$

In words, rad$(S)$ is radius of the smallest closed ball centered in con$(S)$ containing $S$. Next, we will provide a formal definition for the sum of sets. Let $S_1, \ldots, S_m$ be a family of $m$ compact subsets of $\mathbb{R}^N$. The vector sum of $S_1, \ldots, S_m$, denoted by $W$ is:

$$W \equiv \sum_{i=1}^{m} S_i \equiv \left\{ w \colon w = \sum_{i=1}^{m} x^i, x^i \in S_i \right\}$$

**Theorem 1.** (Shapley-Folkman) *Let $S_1, \ldots, S_m$ be a family of $m$ compact subsets of $\mathbb{R}^N$; $W = \sum_{i=1}^m S_i$. Let $L \geq rad(S_i)$ for all $S_i$; let $n = \min(N, m)$. Then for any $x \in con(W)$:*

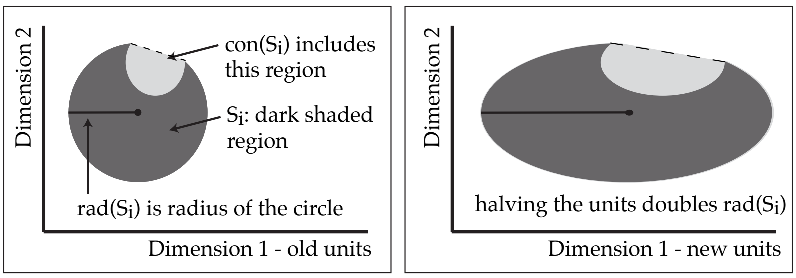
1. *$x = \sum_{i=1}^m x^i$, where $x^i \in con(S_i)$ and with at most $n$ exceptions, $x^i \in S_i$;*
2. *there is $y \in W$ so that $|x - y| \leq L\sqrt{n}$.*

*Proof.* See Starr (1969). □

Roughly speaking, the Shapley-Folkman theorem states that the sum of compact, convex sets is "approximately convex" in the sense that as we add more sets to the sum, the "dimension" of the nonconvexity remains fixed. Let us return to the example above, where each set is a disc with in $\mathbb{R}^2$ with an indentation or hole. Theorem 1 would say that as the size of the sum of sets grows there would be holes with radii no larger than the root of the sum of squares of the two largest in the sum. Since the number of holes stays the same as the set expands, the area from the holes is eventually dwarfed by the total area of the sum of sets.

Given the conversion from vector to scalar via $L$, the bounds generated by Theorem 1 may be sensitive to the choice of units in $\mathbb{R}^N$. To see this visually, consider the very simple example in Figure 1 with $N = 2$. In the left panel, the set of interest $S_i$ is a filled circle with a small hole in it. In this case, $rad(S_i)$ is simply

[1] See also Starr (1969 and 2011).




This simple example depicts a simple non-convex set $S_i \in \mathbb{R}^2$. The panel on the right depicts the same set, where the horizontal axis is measured in units which have been cut in half.

**Figure 1:** The effect of changing units on $rad(S_i)$

the radius of the circle. In the right panel, the units on the horizontal axis have been halved, making $con(S_i)$ ellipse-like and doubling $rad(S_i)$. If we make the units on the horizontal axis arbitrarily small, the shape becomes more elongated and $rad(S_i)$ can be made as large as we wish. Since $L \geq \max_{i=1,\ldots,m} rad(S_i)$, a "poor" choice of units along a single dimension can make the bound from point (2) of Theorem 1 quite uninformative.

We present a simple corollary to the Shapley-Folkman theorem incorporating a normalization which eliminates this sensitivity.

**Corollary 1.** (Corollary to Shapley-Folkman Theorem) *Let $S_1, \ldots, S_m$ be a family of $m$ compact subsets of $\mathbb{R}^N$; $W = \sum_{i=1}^m S_i$ and let $n = \min(N, m)$. Let $b \in \mathbb{R}^N$ where $b_j$, the $j^{th}$ component of $b$, is given by:*

$$b_j = \max \left\{ \left| \sum_{i \in Z} x_j^i \right| : x_1^i \in S_i, Z \subseteq \{1, 2, \ldots, m\}, \#Z = n \right\}$$

*and define*

$$\Omega^* = \begin{pmatrix} 1/b_1 & 0 & \ldots & 0 \\ 0 & 1/b_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1/b_N \end{pmatrix}$$

*Then for any $x \in con(W)$ there is $y \in W$ so that*

$$|\Omega^*(x - y)| \leq \sqrt{n}.$$

*Proof.* This statement is a direct implication of point (1) of the Shapley-Folkman Theorem. The vector $b$ places an upper bound on the magnitude of the sum of any $n$ elements of $S_1, \ldots, S_m$. Therefore, multiplying $x$ and $y$ by $\Omega$ normalizes the units of each coordinate so that the sum of the $n$ largest elements is 1. Given point (1) of the Shapley-Folkman Theorem, the upper bound of this difference will be $\sqrt{n}$. □

## 3. The Anderson (1978) Core Convergence Theorem

As a motivating example, we will consider the Anderson (1978) theorem of core convergence of a pure exchange economy which holds even in the case where the strictly preferred net trade sets of individual households are non-convex.[2] Section 3.1 states the theorem. Section 3.2 motivates our normalization in Anderson's (1978) setting. Section 3.3 restates the theorem with our normalization and demonstrates that our bounds exhibit some desirable properties.

### *3.1. Statement of the Anderson (1978) Theorem*

Consider a pure exchange economy with a set of households, $H$, and $N$ commodities, measured in $\mathbb{R}_+^N$.[3] Each household $i \in H$ has endowment $r^i$. Each household will receive core allocation $x^{oi}$, and we will consider the set of net trades.[4] We need very few assumptions on household preferences, but we do need to assume weak monotonicity, meaning that if $x, y$ are bundles in the household's choice space $X^i \subseteq \mathbb{R}_+^N$, $x \gg y \Rightarrow x \succ_i y$ for all $i$, and free disposal.

**Theorem 2.** (Anderson) *Consider a pure exchange economy with $H$ households and $N$ commodities in $\mathbb{R}_+^N$, with $H > N$. Each household $i \in H$ has endowment $r^i$ and has preference ordering $\succ_i$ on $X^i \subseteq \mathbb{R}_+^N$ that*

[2]Our notation is taken directly from Starr (2011) to be more easily comparable with the previous section.

[3]Our definition of $\mathbb{R}_+^N$ includes the zero vector.

[4]We assume that the reader is familiar with the definition of the core of an exchange economy. For further detail, please refer to Starr (2011), Chapter 21-22.

*satisfies weak monotonicity and free disposal. Define:*

$$M = \max_{n \in \{1,\ldots,N\}} \left\{ \sum_{i \in S} r_n^i \colon S \subseteq H, \#S = N \right\}$$

*where $r_n^i$ is household $i'$s endowment of good $n$. Let $\{x^{oi} \colon i \in H\}$ be a core allocation in this economy. Then if $X^i = \mathbb{R}_+^N$ $\forall i \in H$, there exists $p$ in the unit simplex such that:*

1. $\sum_{i \in H} \left| p \cdot (x^{oi} - r^i) \right| \leq 2M$
2. $\sum_{i \in H} \left| \inf_{x \in \Gamma^i} \{ p \cdot (x - r^i) \} \right| \leq 2M$*, where* $\Gamma^i \equiv \left\{ x \colon x \succ_i x^{oi} \right\}$

*Proof.* See Anderson (1978), Ichiishi (1983), or Starr (2011). □

The first statement suggests that the core allocation approximately fulfills households' competitive equilibrium budget constraints. The second statement says that the core allocation approximately minimizes expenditure subject to a utility constraint.

$M$ is a scalar representing the largest quantity of any single consumption good endowed to an arbitrary collection of $N$ consumers. If the number of households is large, one can effectively treat $M$ as if it is a fixed constant, since the addition of an additional household to the economy is unlikely to change $M$ if $\#H \gg N$. By dividing both sides of each inequality by $\#H$ (the number of households), it follows that as $\#H \to \infty$, the ratio $\frac{2M}{\#H}$ (an upper bound on the average "discrepancy" per household) converges to zero, implying that the core of the pure exchange economy converges to the competitive equilibrium.

If one is concerned only with the limiting result, then it is sufficient to observe that $\frac{2M}{\#H} \to 0$ as $\#H \to \infty$. However, note that this expression is a troublesome to interpret for fixed $H$. Presumably both sides of the inequality are measured in dollars, but given that prices have been normalized to be on the unit simplex, it is very difficult to say whether the magnitude of $2M$ is large relative to the size of the economy. In addition, $M$ could change as the number of households and endowments change, making it very difficult to make comparisons based on this metric.

To address this issue, we suggest dividing the above inequalities by $\sum_{i \in H} p \cdot r^i$, the value of the economy as a whole, as defined by competitive equilibrium prices, rather than $\#H$.[5] This yields an upper bound of $\frac{2M}{\sum_{i \in H} p \cdot r^i}$ which is unitless and can simply be interpreted as the percentage "discrepancy" between the core allocation and approximate competitive equilibrium allocation relative to the value of the economy as a whole. This metric would allow for comparisons across economies with different numbers of agents and/or endowments.

### *3.2. Motivation for Proposed Normalization*

Consider the Anderson (1978) result in a pure exchange economy. While both ratios discussed above will eventually converge to zero, one might want to ask how close they are for a finite number of households, or to compare ratios from one economy with another. As currently defined, the scalar $M$ inherently depends on the units in which we measure the $N$ commodities. To see this, let us define a new vector $b$ as

$$b = (b_1, \ldots, b_N) = \left( \max \left\{ \sum_{i \in S} r_1^i \colon S \subseteq H, \#S = N \right\}, \ldots, \max \left\{ \sum_{i \in S} r_N^i \colon S \subseteq H, \#S = N \right\} \right).$$

It is quite likely that the magnitudes of the coordinates of $b$ vary substantially. For example, $b_1$ could be diamonds in carats, while $b_2$ could be hard drive space in megabytes. Since our bound depends only on the

[5]Note that in some cases competitive equilibrium prices may not exist, or there may be multiple equilibria. The Anderson (1978) theorem guarantees the existence of at least one set of approximate equilibrium prices. Therefore, our unitless measure will always be defined using this approximate price vector. If the core is very large, then different allocations might result in different approximate equilibrium price vectors. In these cases, our unitless measure could be used to provide a range of bounds. We will ignore these issues in our exposition, though our results extend to both cases. See also Anderson (1982) for results on core convergence using a similar, market value-based norm.

largest of these coordinates (let's assume that the $j^{th}$ element of $b$, $b_j$ is the largest), our bound is extremely sensitive to the units in which commodity $j$ is measured.

Now, change the units of commodity $j$ so that the new unit of measurement is 100 times smaller than the original unit. To accomplish this, simply multiply each endowment by a diagonal matrix $\Psi$, equal to the $N \times N$ identity matrix except for a 100 as its $j^{th}$ diagonal element. We will refer to $\Psi$ as a "conversion matrix", since it provides a linear transformation between the new units and the old units. Now $\hat{x}^{oi} = \Psi x^{oi}$ and $\hat{r}^i = \Psi r^i$, so our vector $\hat{b}$ is given by

$$\hat{b} = \Psi b = (b_1, \ldots, 100b_j, \ldots, b_N),$$

where hats have been added to signify that a change of units has been made. Thus, $\hat{M} = 100b_j$. Thus, even though no material change was made to the economy in question, the upper bound increased by a factor of 100. Next, consider the opposite side of the first inequality from the Anderson theorem. We can define the new price vector $\hat{p}$ as:

$$\hat{p} = \frac{1}{\epsilon}\Psi^{-1}p = \frac{1}{\epsilon}(p_1, \ldots, \frac{1}{100}p_j, \ldots, p_n)$$

where $\epsilon \equiv \sum_{k=1}^{N} \psi_k^{-1} p_k$ is a factor which keeps $\hat{p}$ on the simplex and $p_k$ and $\psi_k$ are the $k^{th}$ elements of $p$ and $\Psi$, respectively. It follows that:

$$\sum_{i \in H} \left| p \cdot (x^{oi} - r^i) \right| = \sum_{i \in H} \left| \Psi^{-1} p \cdot \Psi(x^{oi} - r^i) \right| = \epsilon \sum_{i \in H} \left| \hat{p} \cdot (\hat{x}^{oi} - \hat{r}^i) \right|$$

In the extreme case where $p_j = 0$, when applying point (1) of the Anderson theorem in the hat economy, the left hand side of the inequality hasn't changed, even though the right hand side has increased by a factor of 100. It is easy to show that if $p_j > 0$, $\epsilon \leq 100$.[6] This is not surprising, because the price mechanism automatically adjusts for the change in units. Thus, for a finite $H$, the ratios $\frac{2\hat{M}}{\#H}$ and $\frac{2\hat{M}}{\sum_{i \in H} \hat{p} \cdot \hat{r}^i}$ are biased upward by this transformation.

Returning to our previous example, if one of the commodities happened to be hard drive space measured in bytes, one could run into a problem. 1 terabyte ($10^{12}$, or 1 trillion, bytes) of hard drive space costs less than \$100, and one would imagine that someone owns at least 1,000 terabytes of hard drive space. Even if we were to divide this amount of holdings by 10 billion, more than the entire human population on the planet, Theorem 2 would bound the discrepancy at more than \$2 million.[7] Dividing through by the value of all endowments on the planet, we still might have a relatively large upper bound on the magnitude of the discrepancy. Thus, while the price mechanism easily adjusts to changes in units, the bound generated using the Shapley-Folkman theorem is not quite as flexible.

### *3.3. Restatement of Anderson (1978) Theorem with Normalization*

We propose a simple normalization which eliminates the sensitivity of the result to the units in underlying commodity space, and allows us to get as tight a bound on the ratio $\frac{2M}{\sum_{i \in H} p \cdot r^i}$ as possible using Anderson's proof. Let us first introduce a bit more notation.

**Definition.** *Given an initial pure exchange economy $\Sigma$ with $N$ commodities and households $H$, each with endowment $r^i$ and preference relation $\succ_i$ defined on $X^i \subseteq \mathbb{R}^N_+$, a* normalized economy *$\hat{\Sigma}(\Omega)$ with positive definite, diagonal $(N \times N)$ conversion matrix $\Omega$, set of households $\hat{H}$, endowments $\hat{r}^i$, preference relation $\hat{\succ}_i$ is a pure exchange economy that satisfies the following properties:*

1. $\hat{H} = H$.
2. $\hat{r}^i = \Omega r^i$.

[6] If $p_j$ is close to zero, such as when there are many goods with positive prices, $\epsilon \approx 1$.

[7] Note that this number also reflects a normalization of prices to lie on the simplex, so the magnitude of the actual discrepancy is a bit difficult to interpret.

3. *Let $\omega_j$ be the $j^{th}$ diagonal element of $\Omega$. In the normalized economy, the units of good $j$ are divided by $\omega_j$.*
4. *Preferences $\hat{\succ}^i$ are defined on $\Omega X^i$. Given $x, y \in X^i \subseteq \mathbb{R}^N$, if $x \succ_i y$ in $\epsilon$, then $(\Omega x)\hat{\succ}^i(\Omega y)$ in $\hat{\Sigma}(\Omega)$.*

Note that the last item implies that if $x^{oi}$ is a core allocation in $\Sigma$, then $\Omega x^{oi}$ will be a core allocation in $\hat{\Sigma}(\Omega)$. We are now in a position to present a simple corollary to the Anderson (1978) theorem for the normalized economy.

**Corollary 2.** (Corollary to Anderson (1978) Core Convergence Theorem) *Consider a pure exchange economy with households $H$ and $N$ commodities in $\mathbb{R}^N_+$. Each household $i \in H$ has endowment $r^i$ and has preference ordering $\succ_i$ on $X^i \subseteq \mathbb{R}^N_+$ that satisfies weak monotonicity and free disposal. Define:*

$$b = (b_1, \ldots, b_N) = \left( \max \left\{ \sum_{i \in S} r^i_1 \colon S \subseteq H, \#S = N \right\}, \ldots, \max \left\{ \sum_{i \in S} r^i_N \colon S \subseteq H, \#S = N \right\} \right)$$

*where $r^i_n$ is household $i'$s endowment of good $n$. Let $\{x^{oi} \colon i \in H\}$ be a core allocation in this economy. Consider the normalized economy $\hat{\Sigma}(\Omega^*) = \{\hat{H}, \hat{r}^i, \hat{\succ}^i\}$, with conversion matrix:*

$$\Omega^* = \begin{pmatrix} 1/b_1 & 0 & \ldots & 0 \\ 0 & 1/b_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1/b_N \end{pmatrix}$$

*Then if $X^i = \mathbb{R}^N_+ \ \forall i \in H$, there exists $\hat{p}$ in the unit simplex such that for $\hat{x}^{oi} = \Omega^* x^{oi}$:*

1. $\sum_{i \in \hat{H}} \left| \hat{p} \cdot (\hat{x}^{oi} - \hat{r}^i) \right| \leq 2$
2. $\sum_{i \in \hat{H}} \left| \inf_{x \in \hat{\Gamma}^i} \{ \hat{p} \cdot (x - \hat{r}^i) \right| \leq 2$, *where* $\hat{\Gamma}^i \equiv \left\{ x \colon x \hat{\succ}^i \hat{x}^{oi} \right\}$

*Proof:* The proof follows directly from applying the Anderson (1978) theorem to the normalized economy. We simply note that $\hat{x}^{oi}$ is a core allocation in $\hat{\Sigma}(\Omega^*)$ and $\hat{M} = \max_{i=1,\ldots,N} b_i/b_i = 1$ given our choice of conversion matrix. □

To see the advantages of the normalization, we can compare the bounds from Corollary 2 with Theorem 2. Since both sets of inequalities are measured in different units, we multiply both sides of one inequality by a scalar to generate an apples-to-apples comparison where the terms on the left hand side of each inequality are equal.

**Theorem 3.** *Consider a pure exchange economy satisfying the conditions in Corollary 2. Define $b$ and $\Omega^*$ as in Corollary 2 and $M$ as in Theorem 2. Let $\hat{p}$ on the unit simplex satisfy conditions (1) and (2) from Corollary 2 in $\hat{\Sigma}(\Omega^*)$. Let $p = \left( \sum_{k=1}^N b_k^{-1} \hat{p}_k \right)^{-1} \Omega^* \hat{p}$. Then, in the untransformed economy, $p$ satisfies*

1. $\sum_{i \in H} \left| p \cdot (x^{oi} - r^i) \right| \leq 2M$
2. $\sum_{i \in H} \left| \inf_{x \in \Gamma^i} \{ p \cdot (x - r^i) \} \right| \leq 2M$, *where* $\Gamma^i \equiv \{ x \colon x \succ_i x^{oi} \}$

*Proof.* First, we will consider the inequalities from point (1) of each theorem. Let $\hat{p}$ be a price vector which satisfies points (1) and (2) in the normalized economy. Then $p = \left( \sum_{k=1}^N b_k^{-1} \hat{p}_k \right)^{-1} \Omega^* \hat{p}$, which implies (since $\hat{p}$ is on the unit simplex) that $\hat{p} = \left( \sum_{k=1}^N b_k p_k \right)^{-1} \Omega^{*-1} p$. Then, it follows that

$$\sum_{i \in H} \left| p \cdot (x^{oi} - r^i) \right| = \sum_{i \in H} \left| \Omega^{*-1} p \cdot \Omega^* (x^{oi} - r^i) \right| = \left( \sum_{k=1}^N b_k p_k \right) \sum_{i \in \hat{H}} \left| \hat{p} \cdot (\hat{x}^{oi} - \hat{r}^i) \right| \leq 2 \sum_{k=1}^N b_k p_k \leq 2M \sum_{k=1}^N p_k = 2M,$$

where the first inequality holds by assumption, the second inequality holds from the definition of $M$, and the last equality holds because prices $p$ were normalized to the simplex. Note that the last inequality holds strictly if $M > b_k$ and $p_k > 0$ for some $k$.

One obtains an analogous result by following the same steps when comparing the second set of inequalities from point (2) of the respective theorems. $\square$

Theorem 3 states that a suitably transformed version of an arbitrary price vector $\hat{p}$ from the normalized economy of Corollary 2 satisfies Anderson's (1978) bounds in the untransformed economy. Moreover, provided that $M > b_k$ and $p_k > 0$ for some $k$, our normalization generates a tighter bound with respect to the original metric considered in Anderson (1978).

Given the inequality in Corollary 2, one obtains the unitless measure discussed in Section 3.1 by dividing both sides by $\sum_{i\in H} \hat{p}\cdot\hat{r}_i$, rather than $\#H$. As was discussed earlier, we need not make any transformations to make comparisons across potential normalizations. At the cost of introducing a bit more notation, we can show that our normalization provides the tightest possible bound with respect to this measure.

**Theorem 4.** *Consider a pure exchange economy satisfying the conditions in Corollary 2. Define $b$ and $\Omega^*$ as in Corollary 2 and $M$ as in Theorem 2. Let $\hat{p}$ on the unit simplex satisfy conditions (1) and (2) from Corollary 2 in $\hat{\Sigma}(\Omega^*)$. Let $p = \left(\sum_{k=1}^{N} b_k^{-1}\hat{p}_k\right)^{-1}\Omega^*\hat{p}$. Consider the set $U$ of all positive definite, diagonal $(N\times N)$ conversion matrices with finite coefficients. Define*

$$\Theta(A,p) \equiv \frac{2\hat{M}(A)}{\sum_{i\in H}\hat{p}(A,p)\cdot\hat{r}^i(A)},$$

*where $\hat{M}(A) = \max_{n\in\{1,\ldots,N\}}\left\{\sum_{i\in S} a_n r_n^i \colon S\subseteq H, \#S = N\right\}$, $\hat{p}(A) = \frac{A^{-1}p}{\sum_{n=1}^{N} a_n^{-1}p_n}$, and $\hat{r}(A) = Ar^i$, where $a_n$ is the $n^{th}$ diagonal element of $A$ and $p_n$ is the $n^{th}$ element of $p$. Then*

$$\Omega^* \in \underset{A\in U}{\operatorname{argmin}}\,\Theta(A,p).$$

*Proof:* We will perform a proof by contradiction, and the intuition behind the proof is simple. We will assume to the contrary that $\Omega^*$ does not minimize the objective, meaning that there exists some other conversion matrix, $A^*$ which performs better. Then, we will "undo" the change of units caused by $A^*$ and replace it with $\Omega^*$ and show that the objective must decrease. This contradicts the original assumption that $A^*$ was a minimizer, establishing the claim.

Suppose $\Omega^*$ does not minimize $\Theta(A,p)$. Then there exists a diagonal, positive definite matrix $A^*$ with positive coefficients such that $\Psi(A^*) < \Psi(\Omega^*)$. Consider the vector $\tilde{b} = A^*b$, where $b$ is defined as above. Since $\tilde{b}$ has a finite number of elements, there exists $i$ such that $\tilde{b}_i = M(A^*) \geq \tilde{b}_j$ for all $j\in\{1,\ldots,N\}$. Let

$$\Xi = \begin{pmatrix} 1/\tilde{b}_1 & 0 & \ldots & 0 \\ 0 & 1/\tilde{b}_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1/\tilde{b}_N \end{pmatrix},$$

where $\tilde{b}_1, \ldots, \tilde{b}_N$ are the elements of $\tilde{b}$. Note that $\Xi A^* = \Omega^*$. Then

$$\begin{aligned}
\Theta(A^*, p) &= \frac{2\hat{M}(A^*)}{\sum_{i\in H} \hat{p}(A^*, p) \cdot \hat{r}^i(A^*)} = \frac{2\tilde{b}_i}{\sum_{i\in H}(\Xi^{-1}\hat{p}(A^*, p)) \cdot (\Xi\hat{r}^i(A^*))} \\
&= \frac{2}{\sum_{i\in H} \hat{p}(\Omega^*, p) \cdot \hat{r}^i(\Omega^*)} \cdot \frac{\tilde{b}_i}{\sum_{j=1}^{N} \tilde{b}_j \hat{p}_j(A^*, p)} \\
&\geq \frac{2}{\sum_{i\in H} \hat{p}(\Omega^*, p) \cdot \hat{r}^i(\Omega^*)} \cdot \frac{\tilde{b}_i}{\tilde{b}_i \sum_{j=1}^{N} \hat{p}_j(A^*, p)} \\
&= \frac{2}{\sum_{i\in H} \hat{p}(\Omega^*, p) \cdot \hat{r}^i(\Omega^*)} = \Theta(\Omega^*, p)
\end{aligned}$$

where the inequality holds weakly by the definition of $\tilde{b}_i$ and strictly if $\tilde{b}_i > \tilde{b}_j$ for at least one j. The second to last equality holds because we normalized $\hat{p}(A^*, p)$ to reside on the simplex.

Thus we have arrived at a contradiction, so our original assumption that there is a diagonal matrix $A^*$ such that $\Theta(A^*, p) < \Theta(\Omega^*, p)$ must be false. □

$\Theta(\cdot)$ is the upper bound on the unitless measure of discrepancy obtained by applying Theorem 2 in the normalized economy $\hat{\Sigma}(A)$. Thus, Theorem 4 demonstrates that, holding the direction of the price vector fixed, our suggested normalization obtains the least upper bound with respect to the unitless metric. As with Theorem 3, one could follow an analogous set of steps for the unitless measure associated with point (2) of Corollary 2. If there is significant heterogeneity in the magnitudes of the individual elements of $b$, the improvement in the bound is likely to be substantial.

## 4. Conclusion

We observe that the upper bound on the distance between a sum of non-convex sets and its convex hull used in applications of the Shapley-Folkman theorem involves a loss of information which occurs when a vector is converted to a scalar. This effect can be particularly pronounced when there is significant heterogeneity in the units of measurement in the underlying space. We propose a simple normalization which eliminates this loss of information and the sensitivity to the choice of units. This normalization is simple to implement, since it merely requires performing one additional step prior to applying the Shapley-Folkman theorem.

As a motivating example, we apply our normalization to Anderson's (1978) classical result on core convergence in a pure exchange economy. We present an alternative, unitless measure of the discrepancy between an arbitrary core allocation and the corresponding competitive equilibrium allocation, and demonstrate that our normalization generates a least upper bound (using the Anderson result) with respect to this measure. This measure allows for comparisons across economies with different numbers of agents and different endowments and a more intuitive interpretation of numerical values.

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